

Asymptotically Free Yang-Mills Classical Mechanics with Self-Linked Orbits

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Abstract

We construct a classical mechanics Hamiltonian which exhibits spontaneous symmetry breaking akin the Coleman-Weinberg mechanism, dimensional transmutation, and asymptotically free self-similarity congruent with the beta-function of four dimensional Yang-Mills theory. Its classical equations of motion support stable periodic orbits and in a three dimensional projection these orbits are self-linked into topologically nontrivial, toroidal knots.

The non-perturbative structure of four dimensional Yang-Mills theory continues to be the subject of extensive investigations. A major goal is the understanding of large distance properties such as color confinement, mass gap and the glueball spectrum. The Yang-Mills theory has also a number of well established salient features like ultraviolet asymptotic freedom and the presence of finite action instantons. Here we shall introduce a *classical* mechanics Hamiltonian which contains many ingredients of the four dimensional Yang-Mills field theory, even though it is defined in a four dimensional *phase* space. These include asymptotically free self-similarity with a coupling constant that flows like the one loop coupling constant of four dimensional Yang-Mills theory, dimensional transmutation, and spontaneous symmetry breaking akin the Coleman-Weinberg mechanism. Furthermore, we find that its Hamilton's equations support stable periodic

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orbits. Remarkably, we find that in a three dimensional projection these orbits turn out to be self-linked into toroidal knots. As usual, their self-linking number can then be computed by a three dimensional Chern-Simons functional [1]. This can be viewed as another trait of four dimensional Yang-Mills theory.

We motivate our classical mechanics model by considering the infrared four dimensional $SU(2)$ Yang-Mills theory, in a limit where all spatial inhomogeneities can be ignored. In the maximal abelian gauge the classical Yang-Mills action is then approximated by a classical mechanics action [2]

$$S = \int_0^T d\tau \{ p_i \partial_\tau q_i - \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}(q_1^2 - q_2^2)^2 \} \quad (1)$$

where $\tau = t/t_0$ is dimensionless, and we have also scaled p_i and q_i into dimensionless quantities. When we rotate the coordinates by $\pi/4$ this becomes the standard $x^2 y^2$ action [3]. This is a quite universal model and besides the Yang-Mills theory it relates *e.g.* to the low energy limit of (super)membranes [4]. Despite its apparent simplicity the $x^2 y^2$ model has a number of remarkable properties [3]. Most notably, the classical dynamics of (1) is chaotic which originally led to a conjecture that the model is ergodic. This was proven to be wrong by [5], who found that the phase space of (1) admits a non-ergodic island consisting of a periodic orbit surrounded by a stable, invariant torus of classical solutions. Since the quantum mechanical partition function is evaluated by the Gutzwiller trace formula [6] which sums over all periodic solutions to Hamilton's equations, these stable periodic orbits are clearly pivotal to the quantization of (1). In fact, one could argue that if (1) indeed does approximate the infrared limit of the $SU(2)$ Yang-Mills theory, the energies of these stable periodic orbits are avatars of stable glueball states in the Yang-Mills quantum theory.

In the present Letter we are interested in improving the relations between (1) and the pure four dimensional $SU(2)$ Yang-Mills theory, maybe even the (super)membrane theory. Our starting point is the one loop Yang-Mills effective action, originally computed in [7]. Recently this computation has been revisited in [8], employing a decomposed [9] Yang-Mills field. A comparison of (1) with the results in [8] suggests that we introduce the following improvement,

$$S = \int_0^T d\tau \{ p_i \partial_\tau q_i - \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{2}(q_1^2 - q_2^2)^2 [1 + \frac{\lambda}{2} \ln(q_1^2 - q_2^2)^2] \} \quad (2)$$

When $\lambda = 0$ we return to (1) and the potential has a degenerate minimum with a vanishing energy $E = 0$ along the lines $q_1 = \pm q_2$. For $\lambda > 0$ this minimum becomes unstable, there is a symmetry breaking akin the Coleman-Weinberg mechanism and the new energy minimum occurs along the four branches of the hyperbola $|q_1^2 - q_2^2| =$

$\exp(-\frac{1}{\lambda} - \frac{1}{2})$. Consequently the new minimum is also degenerate but with an additional four-fold discrete multiplicity, and the new minimum energy is

$$E_{min} = -\frac{\lambda}{4} \exp(-1 - \frac{2}{\lambda}) \quad (3)$$

Notice the non-analytic, non-perturbative dependence on the coupling in the small- λ limit. We also note that the minima of the $\lambda = 0$ potential energy of (1) along the lines $q_1 = \pm q_2$ become local maxima of the $\lambda > 0$ potential energy of (2), separating the four branches of the $\lambda > 0$ minima from each other. See figure 1.

We are interested in the periodic classical solutions to the Hamilton's equations of (2), since these are the basic ingredients in the Gutzwiller trace formula quantization. We start by revealing their self-similar scaling properties, in a manner reminiscent of conventional renormalization group transformations. For this we consider a periodic solution $\Gamma(\lambda, T)$ to the Hamilton's equations, with some definite values of the period T and coupling λ . We then inquire how to scale this periodic solution to other periods $T \rightarrow \tilde{T}$, possibly with a redefinition of the coupling $\lambda \rightarrow \tilde{\lambda}$. For this we introduce a change of variables in (2) which scales $T \rightarrow \tilde{T}$, redefines $\lambda \rightarrow \tilde{\lambda}$ and possibly multiplies the action by some overall constant $S \rightarrow \kappa S$ but leaves the functional form of the integrand in (2) otherwise intact. The form of Hamilton's equations remains then untouched but a periodic orbit $\Gamma(\lambda, T)$ becomes mapped into a new periodic orbit $\Gamma(\tilde{\lambda}, \tilde{T})$ solving the original equations with the new parameter values $\tilde{\lambda}, \tilde{T}$.

To implement this scaling transformation explicitly, we first consider the scaling of periodic trajectories in the $x^2 y^2$ model (1),

$$\begin{aligned} t &\longrightarrow c^{-1}t \\ q &\longrightarrow cq \\ p &\longrightarrow c^2 p \end{aligned} \quad (4)$$

This transformation leaves the equations of motion in (1) invariant, and trajectories with period T are mapped into trajectories with period $c^{-1}T$.

In the case of (2), the scaling (4) fails to leave the functional form of the equations of motion intact. For this, we need to modify the scaling of t and p as follows:

$$\begin{aligned} t &\longrightarrow \left(\frac{1-\lambda \ln c^2}{c^2} \right)^{\frac{1}{2}} t \\ p &\longrightarrow \left(\frac{c^4}{1-\lambda \ln c^2} \right)^{\frac{1}{2}} p \end{aligned} \quad (5)$$

and in addition we must redefine

$$\lambda \longrightarrow \frac{\lambda}{1-\lambda \ln c^2} \quad (6)$$

This is then a self-similarity transformation which leaves the overall functional form of the Hamilton's equations of (2) intact; The only effect of this improved scaling transformation in (2) is a renormalization of the coupling constant λ according to (6), a scaling of period T and an overall multiplicative redefinition of the action.

Note that the coupling constant flow (6) is like the flow of the one-loop coupling constant in four dimensional Yang-Mills theory. In particular, in the limit of small c the periodic classical solutions of (2) exhibit asymptotically free self-similarity, in the sense that in this limit (2) approaches (1).

When we increase c we find an upper bound $c_{max}^2 = \exp(1/\lambda)$, where the renormalized coupling constant (6) diverges. In the present case this Landau pole is physically relevant. It determines a critical value of c which sets a lower bound for the period T_{min} , below which a periodic orbit can not be extended by the self-similarity transformation.

The Hamilton's equations of (2) can be readily solved by numerical integrations. Previously it has been shown [5] that with $\lambda = 0$ there is a periodic solution which is surrounded by an stable, invariant torus formed by classical solutions. Subsequently the existence of additional periodic solutions with accompanying invariant torii has been reported *e.g.* in [10]. We have investigated the effects of a nonvanishing λ in (2) to the stability of the solution presented in [5]. When we increase λ from $\lambda = 0$ while keeping the total energy of the solution intact, we find that the torus of classical solutions which surrounds the periodic orbit of [5] retains its invariant character for small values of the coupling λ . But when the coupling approaches a critical value $\lambda_c \approx 0.6$ we find that the torus loses its invariance properties essentially by period doubling. This indicates that at λ_c the periodic orbit of [5] loses its stability.

However, for non-vanishing λ we also find *novel* periodic solutions with the physically interesting property that their energies are negative, $E < 0$. Consequently these solutions have an energy which is *lower* than that of the $\lambda = 0$ ground state. These solutions are also surrounded by stability islands which are formed by invariant torii of $E < 0$ (not necessarily periodic) classical solutions. Since $E < 0$ these stability islands consist of trajectories which are entirely confined in one of the four energy valleys which surround the minimum energy hyperbola $|q_1^2 - q_2^2| = \exp(-\frac{1}{\lambda} - \frac{1}{2})$. Consequently we can describe their properties by restricting to one of the four valleys, and for definiteness we shall select the valley where $q_1 > 0$. We also employ the self-similar scaling property of the action (2) to set $\lambda = 1$. This is akin a dimensional transmutation, where instead of λ the solutions are characterized by some other distinguishing parameter. For this parameter we choose the maximal value of q_1 which is attainable to the trajectory, say at $q_2 = 0$. Notice that since the energy is the sole conserved quantity in (2), the actual motion described by a classical solution occurs in a three dimensional space. Consequently we can visualize the trajectories in terms of a three dimensional projection, and for this we select the subspace (q_2, p_1, p_2) . In particular, we can characterize the trajectories using invariants of this three dimensional space such as their self-linking number, provided the trajectories are indeed linked.

By numerical integration we find that in the $q_1 > 0$ valley the Hamilton's equations of (2) describe a periodic orbit which is stable (when $\lambda = 1$) in the interval $E_{min} \approx -0.0045 \leq E \leq E_{max} \approx -0.0008$. In figure 2 we draw the solution for $E = -0.0010$ in the (q_1, q_2) plane. In figure 3 we plot the Poincare map of the stable solution and

some trajectories in the surrounding invariant torus at the $q_2 = 0$ surface of section; The stability of the solution and its invariant torus is evident. In figure 4 we describe how this solution scales, by plotting the the energy of the trajectory as a function of the maximal value of q_1 ; Notice that by dimensional transmutation this corresponds to changing λ in the original model.

We find that the periodic orbit still exists when the energy E of the trajectory exceeds a critical value $E_{max} \approx -0.0008$. But when E exceeds E_{max} the solutions in the torus loose their stability and the torus shrinks away. At that point our periodic orbit loses its stability.

Finally, we have performed a detailed investigation of the solutions that form the invariant torus. For rational windings around the torus, these solutions are themselves closed orbits. Their geometrical shape can be visualized in the three dimensional (q_2, p_1, p_2) subspace, where the orbits become closed curves in R^3 . Remarkably, we find that these curves can be self-linked into topologically nontrivial torus knots. As an example, in figure 5 we describe a closed orbit which we identify as a figure-eight knot in the three dimensional subspace. Indeed, we propose that *all* torus knot can appear as stable closed orbit solutions to the Hamilton's equations of (2). This is reminiscent of the knots [11] in a model that relates to the four dimensional Yang-Mills theory [9].

In conclusion, we have studied a classical mechanics model which relates both to the four dimensional Yang-Mills theory and the (super)membrane. We have found that its periodic classical solutions exhibit self-similarity with a coupling constant flowing like the Yang-Mills coupling constant. Surprisingly, we have found that such periodic solutions exist even for energies which are lower than the energy of the ground state at vanishing coupling. Furthermore, these solutions are stable in the sense that they are surrounded by an invariant torus which is formed by classical solutions. In this torus we then identify several additional periodic solutions, some of which form nontrivial torus knots. If our model indeed relates to the low energy limit of four dimensional Yang-Mills theory these solutions could be the avatars of its glueball states.

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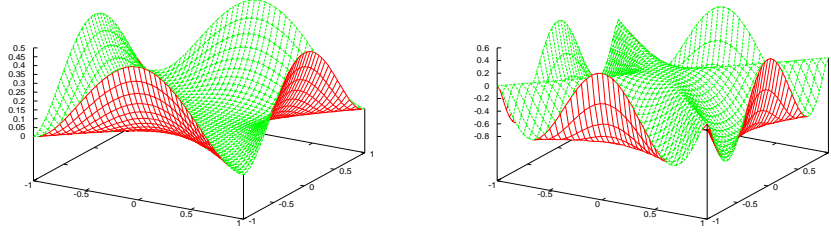


Figure 1: The potential in (2) for $\lambda = 0$ and $\lambda = 10$ respectively.

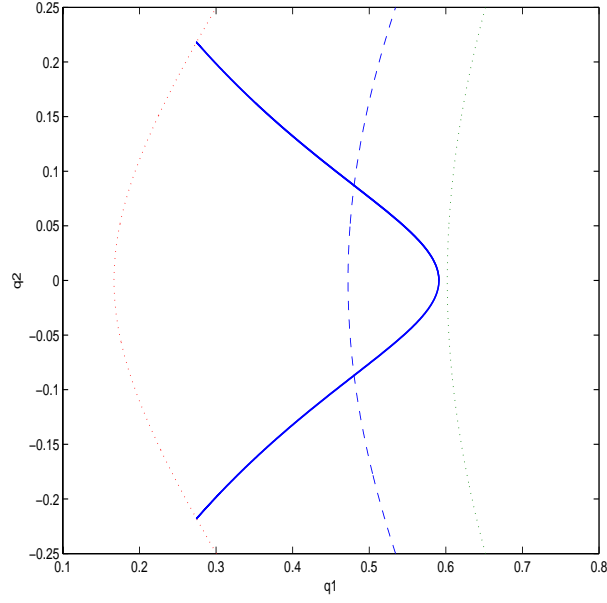


Figure 2: The stable trajectory at λ_1 and $E = -0.001$, drawn on the (q_1, q_2) plane. The dotted lines denote where the potential is $V = -0.001$ and the dashed line is the minimum of the potential.

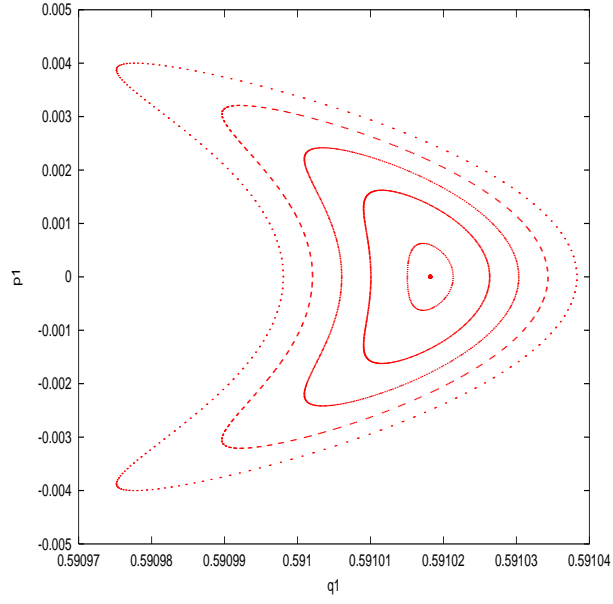


Figure 3: Poincare section at $q_2 = 0$ which describes the stable solution at the center and selected trajectories of the surrounding invariant torus.

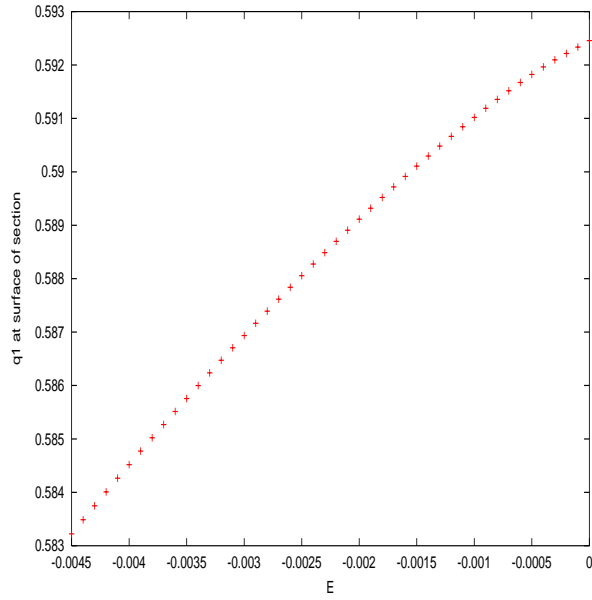


Figure 4: The relation between energy and maximal value of q_1 , at $q_2 = 0$.

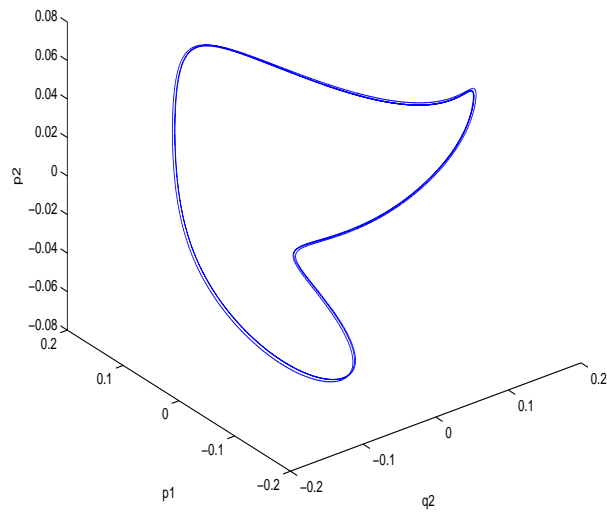


Figure 5: The figure-8 solution in the (q_2, p_1, p_2) subspace. Notice that the torus around which the solution wraps has a very small radius in comparison to its length.